MATHAI_21_HL_Summer_2021_Q1

Solution

1. Value of Daisy's investment after 2 years

The investment follows the **compound interest** formula:

$$A = P\left(1 + \frac{r}{100k}\right)^{kt}$$

where $P = 37000\,\text{AUD}$, $r = 6.4$, $k = 4$ (quarterly), and $t = 2$ years.

- Substitute the values into the formula:

$$\begin{aligned} A &= 37000\left(1 + \frac{6.4}{100 \times 4}\right)^{4\times 2} \\ &= 37000(1.016)^8 \\ &= 42018.173... \end{aligned}$$

  Rounding to the nearest dollar:

$$\boxed{42018\,\text{AUD}}$$

2. Minimum value of m for the investment to exceed 50,000 AUD

We need to find the smallest integer m (months) such that $A > 50000$. Since interest is compounded quarterly, we let n be the number of quarters.

$$37000(1.016)^n > 50000$$

- Solve for n using logarithms:

$$\begin{aligned} (1.016)^n &> \frac{50000}{37000} \\ n\ln(1.016) &> \ln\left(\frac{50}{37}\right) \\ n &> \frac{\ln(50/37)}{\ln(1.016)} \\ n &> 18.94... \end{aligned}$$

  Since n must be an integer for the interest to be credited, $n = 19$ quarters.

- Convert quarters to months:

$$m = 19 \times 3 = 57$$

$$\boxed{57}$$

3. Amount of the loan

The apartment price is $200000\,\text{AUD}$. Daisy pays 25% as an initial payment.

- Initial payment: $0.25 \times 200000 = 50000\,\text{AUD}$.

- Loan amount: $200000 - 50000 = 150000\,\text{AUD}$.

$$\boxed{150000\,\text{AUD}}$$

4. Loan details

(i) Amount of interest paid The loan is for 10 years with monthly payments of $1700\,\text{AUD}$.

- Total payments: $1700 \times 12 \times 10 = 204000\,\text{AUD}$.
- Interest paid: $204000 - 150000 = 54000\,\text{AUD}$.

$$\boxed{54000\,\text{AUD}}$$

(ii) Annual interest rate of the loan Using the **annuity** formula for a loan:

$$PV = PMT\left[\frac{1-(1+i)^{-n}}{i}\right]$$

where $PV = 150000$, $PMT = 1700$, $n = 120$. We solve for the monthly rate i. Using a financial solver or numerical methods:

$$150000 = 1700\left[\frac{1-(1+i)^{-120}}{i}\right] \implies i \approx 0.005439...$$

- Annual interest rate $r = i \times 12 \times 100$:

$$r \approx 6.527...\%$$

$$\boxed{6.53\%}$$

5. Final payment after 5 years

The final payment is the **remaining balance** of the loan after 5 years (60 payments). The balance B is the present value of the remaining 60 payments:

$$B = 1700\left[\frac{1-(1+0.005439...)^{-60}}{0.005439...}\right]$$
$$= 86895.02...$$

Rounding to the nearest dollar:

$$\boxed{86895\,\text{AUD}}$$

6. Money saved by making the final payment

- Total cost if paid over 10 years: $204000\,\text{AUD}$.
- Total cost with early payoff:
  - Payments made in first 5 years: $1700 \times 60 = 102000\,\text{AUD}$.
  - Final payment: $86895\,\text{AUD}$.
  - Total: $102000 + 86895 = 188895\,\text{AUD}$.
- Savings: $204000 - 188895 = 15105\,\text{AUD}$.

$$\boxed{15105\,\text{AUD}}$$

MATHAI_21_HL_Summer_2021_Q2

Solution

The cross-sectional view of the tunnel is modeled by the cubic function:

$$y = -0.1x^3 + 0.8x^2, \quad 2 \leq x \leq 8$$

where y represents the height of the tunnel in metres and x is the horizontal distance from the origin O.

1. Differentiation and Maximum Height

- (i) To find the derivative $\frac{dy}{dx}$, we apply the **power rule** for differentiation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(-0.1x^3 + 0.8x^2) \\ &= -0.1(3x^2) + 0.8(2x) \\ &= -0.3x^2 + 1.6x \end{aligned}$$

- (ii) To find the maximum height, we set the derivative to zero to find the **critical points**:

$$\begin{aligned} -0.3x^2 + 1.6x &= 0 \\ x(-0.3x + 1.6) &= 0 \end{aligned}$$

This gives $x = 0$ (outside the domain) or:

$$\begin{aligned} -0.3x + 1.6 &= 0 \\ 0.3x &= 1.6 \\ x &= \frac{1.6}{0.3} = \frac{16}{3} \approx 5.333 \end{aligned}$$

Substituting $x = \frac{16}{3}$ back into the original equation for y:

$$\begin{aligned} y_{\max} &= -0.1\left(\frac{16}{3}\right)^3 + 0.8\left(\frac{16}{3}\right)^2 \\ &= -0.1\left(\frac{4096}{27}\right) + 0.8\left(\frac{256}{9}\right) \\ &= -\frac{409.6}{27} + \frac{204.8}{9} \\ &= -\frac{409.6}{27} + \frac{614.4}{27} \\ &= \frac{204.8}{27} \approx 7.585\,\text{m} \end{aligned}$$

2. Height at Specific Points

- (i) When $x = 4$:

$$\begin{aligned} y &= -0.1(4)^3 + 0.8(4)^2 \\ &= -0.1(64) + 0.8(16) \\ &= -6.4 + 12.8 = 6.4\,\text{m} \end{aligned}$$

- (ii) When $x = 6$:

$$y = -0.1(6)^3 + 0.8(6)^2$$
$$= -0.1(216) + 0.8(36)$$
$$= -21.6 + 28.8 = 7.2\,\text{m}$$

3. Trapezoidal Rule Estimation

To estimate the cross-sectional area using the **trapezoidal rule** with three intervals over the domain $[2, 8]$, the interval width is $h = \frac{8-2}{3} = 2$. The x-values are $x_0 = 2, x_1 = 4, x_2 = 6, x_3 = 8$. The corresponding y-values are:

- $y(2) = -0.1(8) + 0.8(4) = 2.4$
- $y(4) = 6.4$
- $y(6) = 7.2$
- $y(8) = -0.1(512) + 0.8(64) = -51.2 + 51.2 = 0$

$$\text{Area} \approx \frac{h}{2}[y(2) + 2(y(4) + y(6)) + y(8)]$$
$$= \frac{2}{2}[2.4 + 2(6.4 + 7.2) + 0]$$
$$= 2.4 + 2(13.6)$$
$$= 2.4 + 27.2 = 29.6\,\text{m}^2$$

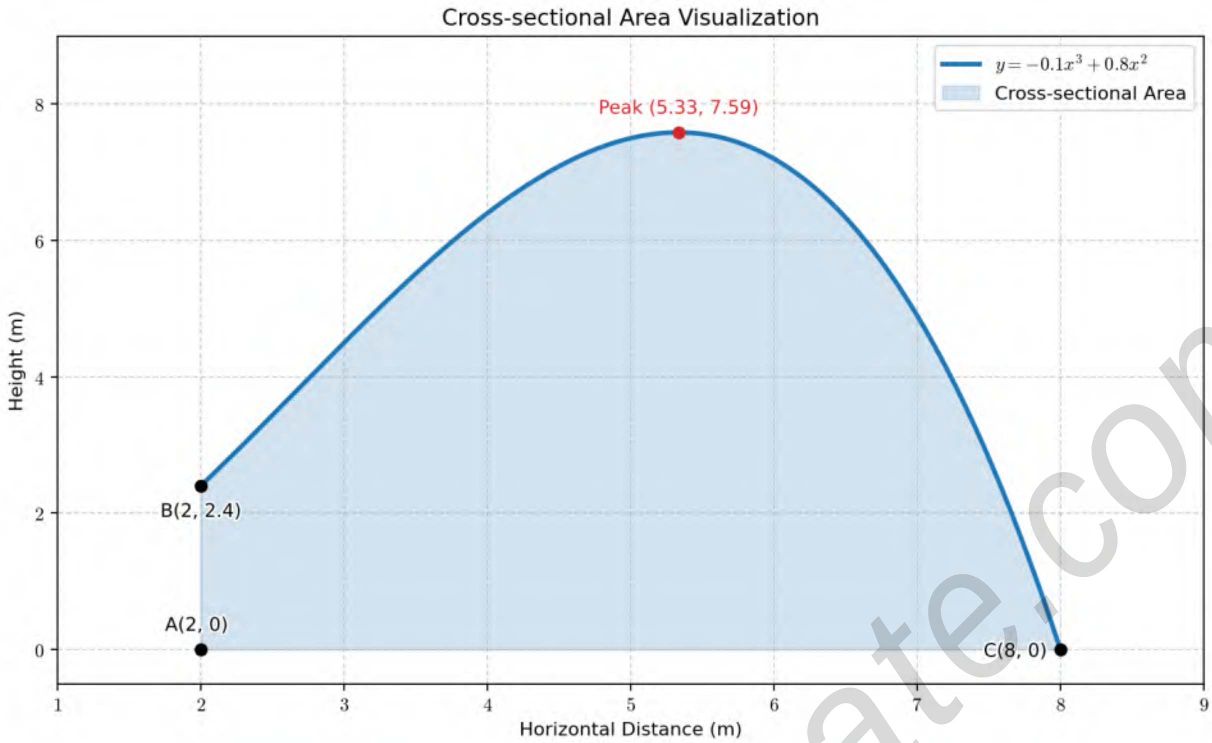
4. Exact Cross-sectional Area

- (i) The integral representing the area under the curve from $x = 2$ to $x = 8$ is:

$$\int_2^8 (-0.1x^3 + 0.8x^2)\,dx$$

- (ii) Evaluating the **definite integral**:

$$\text{Area} = \left[-0.1\frac{x^4}{4} + 0.8\frac{x^3}{3}\right]_2^8$$
$$= \left[-0.025x^4 + \frac{0.8}{3}x^3\right]_2^8$$
$$= \left(-0.025(8)^4 + \frac{0.8}{3}(8)^3\right) - \left(-0.025(2)^4 + \frac{0.8}{3}(2)^3\right)$$
$$= \left(-102.4 + \frac{409.6}{3}\right) - \left(-0.4 + \frac{6.4}{3}\right)$$
$$= -102 + \frac{403.2}{3}$$
$$= -102 + 134.4 = 32.4\,\text{m}^2$$



(a) (i) $\frac{dy}{dx} = -0.3x^2 + 1.6x$

(b) (ii) Maximum height $\approx \boxed{7.59\,\text{m}}$

(c) (i) $x = 4, y = \boxed{6.4\,\text{m}}$

(d) (ii) $x = 6, y = \boxed{7.2\,\text{m}}$

(e) Estimated Area = $\boxed{29.6\,\text{m}^2}$

(f) (i) $\int_2^8 (-0.1x^3 + 0.8x^2)\,dx$

(g) (ii) Exact Area = $\boxed{32.4\,\text{m}^2}$

MATHAI_21_HL_Summer_2021_Q3

Solution

1. Normal Distribution Probabilities

Let X be the random variable representing the stopping distance of a bicycle. It is given that X follows a **Normal distribution** with mean $\mu = 6.76\,\mathrm{m}$ and standard deviation $\sigma = 0.12\,\mathrm{m}$, denoted as $X \sim N(6.76, 0.12^2)$.

- **(i) Probability that a bicycle stops in less than $6.5\,\mathrm{m}$:** We calculate the **z-score** for $x = 6.5$:

$$z = \frac{x-\mu}{\sigma} = \frac{6.5-6.76}{0.12} = -2.16666666666667$$

  Using the **cumulative distribution function** (CDF) of the standard normal distribution $\Phi(z)$:

$$P(X < 6.5) = \Phi(-2.16666666666667) \approx 0.0151301$$

  Correct to four decimal places: $\boxed{0.0151}$

- **(ii) Probability that a bicycle stops in more than $7\,\mathrm{m}$:** We calculate the z-score for $x = 7$:

$$z = \frac{7-6.76}{0.12} = 2.0$$

  The probability is:

$$P(X > 7) = 1 - \Phi(2.0) \approx 1 - 0.9772498 = 0.0227501$$

  Correct to four decimal places: $\boxed{0.0228}$

2. Expected Frequencies

For a sample of $n = 1000$ bicycles, the **expected number** E in an interval $[a, b]$ is given by $E = n \cdot P(a \le X \le b)$.

- **(i) Expected number between $6.5\,\mathrm{m}$ and $6.75\,\mathrm{m}$:**

$$\begin{aligned} P(6.5 \le X \le 6.75) &= P(X \le 6.75) - P(X \le 6.5) \\ &= \Phi\left(\frac{6.75-6.76}{0.12}\right) - 0.0151301 \\ &= \Phi(-0.08333333) - 0.0151301 \\ &\approx 0.4667932 - 0.0151301 = 0.4516631 \end{aligned}$$

  Expected number: $1000 \times 0.4516631 = 451.6631$. Correct to four significant figures: $\boxed{451.7}$

- **(ii) Expected number between $6.75\,\mathrm{m}$ and $7\,\mathrm{m}$:**

$$\begin{aligned} P(6.75 \le X \le 7) &= P(X \le 7) - P(X \le 6.75) \\ &= (1 - 0.0227501) - 0.4667932 \\ &= 0.9772499 - 0.4667932 = 0.5104567 \end{aligned}$$

Expected number: $1000 \times 0.5104567 = 510.4567$. Correct to four significant figures: $\boxed{510.5}$

3. Chi-squared Goodness of Fit Test

- **(c) Null and Alternative Hypotheses:** H_0: The stopping distances follow a normal distribution with mean $6.76\,\mathrm{m}$ and standard deviation $0.12\,\mathrm{m}$. H_1: The stopping distances do not follow a normal distribution with mean $6.76\,\mathrm{m}$ and standard deviation $0.12\,\mathrm{m}$.
- **(d) Find the p-value:** We first summarize the observed (O_i) and expected (E_i) frequencies:

  1. Less than 6.5: $O_1 = 12$, $E_1 = 1000 \times 0.0151301 = 15.1301$
  2. 6.5 to 6.75: $O_2 = 428$, $E_2 = 451.6631$
  3. 6.75 to 7: $O_3 = 527$, $E_3 = 510.4567$
  4. More than 7: $O_4 = 33$, $E_4 = 1000 \times 0.0227501 = 22.7501$

  The **Chi-squared statistic** χ^2 is calculated as:

$$\begin{aligned}\chi^2 &= \sum \frac{(O_i - E_i)^2}{E_i} \\ &= \frac{(12-15.1301)^2}{15.1301} + \frac{(428-451.6631)^2}{451.6631} + \frac{(527-510.4567)^2}{510.4567} + \frac{(33-22.7501)^2}{22.7501} \\ &\approx 0.6475 + 1.2397 + 0.5362 + 4.6180 = 7.0414\end{aligned}$$

  The **degrees of freedom** $df = k - 1$, where k is the number of categories. Since the parameters μ and σ were given (not estimated from the sample), $df = 4 - 1 = 3$. Using the χ^2 distribution with $df = 3$, the **p-value** is:

$$p = P(\chi_3^2 > 7.0414) \approx 0.07052$$

  $\boxed{p \approx 0.0705}$

- **(e) Conclusion:** Since the $p\text{-value} \approx 0.0705$ is greater than the **significance level** $\alpha = 0.05$, we fail to reject the null hypothesis H_0. **Reason:** $0.0705 > 0.05$. There is insufficient evidence to suggest that the stopping distances do not follow the specified normal distribution.

MATHAI_21_HL_Summer_2021_Q4

Solution

1. Linear Model for the Base

The section from $(0, 3.5)$ to $(4, 6)$ is modeled as a straight line.

- The gradient m is calculated as:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 3.5}{4 - 0} = \frac{2.5}{4} = 0.625$$

- Since the line passes through $(0, 3.5)$, the y-intercept c is 3.5.
- The equation is:

$$\boxed{y = 0.625x + 3.5}$$

2. Least Squares Regression Quadratic

(i) Using the points $(4, 6)$, $(6.5, 4)$, $(7, 3)$, and $(7.5, 0)$, we perform a **least squares regression** for a quadratic of the form $y = ax^2 + bx + c$.

- The resulting regression equation is:

$$\boxed{y = -0.536x^2 + 4.45x - 3.23}$$

(ii) To evaluate the model at $x = 4$, we find the gradient dy/dx:

$$\frac{dy}{dx} = 2(-0.536)x + 4.45 = -1.072x + 4.45$$

- At $x = 4$:

$$\frac{dy}{dx}|_{x=4} = -1.072(4) + 4.45 = -4.288 + 4.45 = 0.162$$

- The gradient of the linear section is 0.625. Since $0.162 \neq 0.625$, there is a “sharp corner” at the join. Furthermore, for $(4, 6)$ to be a peak (as suggested by the cupcake shape), the gradient should ideally be 0 or negative immediately after $x = 4$. A positive gradient at $x = 4$ implies the curve is still rising, which contradicts the visual evidence of the cupcake’s top.

3. Vertex Form Quadratic Model

Charlotte proposes a quadratic with a maximum at $(4, 6)$ passing through $(7.5, 0)$.

- Using the **vertex form** $y = a(x - h)^2 + k$ with vertex $(h, k) = (4, 6)$:

$$y = a(x-4)^2 + 6$$

- Substitute the point $(7.5, 0)$ to find a:

$$\begin{aligned} 0 &= a(7.5-4)^2 + 6 \\ -6 &= a(3.5)^2 \\ -6 &= 12.25a \\ a &= -\frac{6}{12.25} = -\frac{24}{49} \approx -0.48979... \end{aligned}$$

- The equation is:

$$\boxed{y = -\frac{24}{49}(x-4)^2 + 6}$$

4. Volume of Revolution

(i) The **volume of revolution** about the x-axis is given by $V = \pi \int y^2\, dx$. The total volume is the sum of the volumes of the two sections:

$$\boxed{V = \pi \int_0^4 (0.625x + 3.5)^2\, dx + \pi \int_4^{7.5} \left(-\frac{24}{49}(x-4)^2 + 6\right)^2 dx}$$

(ii) Calculating the integrals:

- For the first integral (linear part):

$$\begin{aligned} V_1 &= \pi \int_0^4 (0.390625x^2 + 4.375x + 12.25)\, dx \\ &= \pi [0.130208x^3 + 2.1875x^2 + 12.25x]_0^4 \\ &= \pi(8.3333 + 35 + 49) = 92.3333\pi \end{aligned}$$

- For the second integral (quadratic part): Let $u = x - 4$, then $du = dx$. The limits change from $[4, 7.5]$ to $[0, 3.5]$.

$$\begin{aligned} V_2 &= \pi \int_0^{3.5} \left(-\frac{24}{49}u^2 + 6\right)^2 du \\ &= \pi \int_0^{3.5} \left(\frac{576}{2401}u^4 - \frac{288}{49}u^2 + 36\right) du \\ &= \pi \left[\frac{576}{12005}u^5 - \frac{96}{49}u^3 + 36u\right]_0^{3.5} \\ &= \pi(25.2105 - 84 + 126) = 67.2105\pi \end{aligned}$$

- Total Volume:

$$\begin{aligned} V_{\text{total}} &= (92.3333 + 67.2105)\pi \\ &= 159.5438\pi \approx 501.218 \end{aligned}$$

Rounding to three significant figures:

$$\boxed{501\,\text{cm}^3}$$

MATHAI_21_HL_Summer_2021_Q5

Solution

The problem describes a **Markov chain** with two states: Sunny (S) and Not Sunny (N). The transition matrix T is given by:

$$T = (0.8 \;\; 0.3; 0.2 \;\; 0.7)$$

where the first row/column corresponds to "Sunny" and the second to "Not Sunny".

1. Probability of sunny weather in three days Given that it is sunny today, the initial state vector is $X_0 = \binom{1}{0}$. The state vector after three days, X_3, is calculated using the relation $X_n = T^n X_0$. - Calculate T^2:

$$\begin{aligned} T^2 &= (0.8 \;\; 0.3; 0.2 \;\; 0.7)(0.8 \;\; 0.3; 0.2 \;\; 0.7) \\ &= ((0.8 \times 0.8) + (0.3 \times 0.2) \;\; (0.8 \times 0.3) + (0.3 \times 0.7); (0.2 \times 0.8) + (0.7 \times 0.2) \;\; (0.2 \times 0.3) + (0.7 \times 0.7)) \\ &= (0.7 \;\; 0.45; 0.3 \;\; 0.55) \end{aligned}$$

- Calculate T^3:

$$\begin{aligned} T^3 = T \cdot T^2 &= (0.8 \;\; 0.3; 0.2 \;\; 0.7)(0.7 \;\; 0.45; 0.3 \;\; 0.55) \\ &= ((0.8 \times 0.7) + (0.3 \times 0.3) \;\; (0.8 \times 0.45) + (0.3 \times 0.55); (0.2 \times 0.7) + (0.7 \times 0.3) \;\; (0.2 \times 0.45) + (0.7 \times 0.55)) \\ &= (0.65 \;\; 0.525; 0.35 \;\; 0.475) \end{aligned}$$

- The state vector X_3 is:

$$X_3 = T^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.65 \\ 0.35 \end{pmatrix}$$

The probability that it will be sunny in three days is the first component of X_3. $\boxed{0.65}$

2. Eigenvalues and Eigenvectors of T To find the **eigenvalues** λ, we solve the **characteristic equation** $\det(T - \lambda I) = 0$:

$$\begin{aligned} \det(0.8 - \lambda \;\; 0.3; 0.2 \;\; 0.7 - \lambda) &= (0.8 - \lambda)(0.7 - \lambda) - (0.3 \times 0.2) \\ &= \lambda^2 - 1.5\lambda + 0.56 - 0.06 \\ &= \lambda^2 - 1.5\lambda + 0.5 = 0 \end{aligned}$$

Solving the quadratic equation $(\lambda - 1)(\lambda - 0.5) = 0$, we get $\lambda_1 = 1$ and $\lambda_2 = 0.5$.

- For $\lambda_1 = 1$:

$$(-0.2 \;\; 0.3; 0.2 \;\; -0.3)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies -0.2x + 0.3y = 0 \implies 2x = 3y$$

  An **eigenvector** is $v_1 = \binom{3}{2}$ (or any scalar multiple).

- For $\lambda_2 = 0.5$:

$$(0.3 \;\; 0.3; 0.2 \;\; 0.2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x + y = 0 \implies x = -y$$

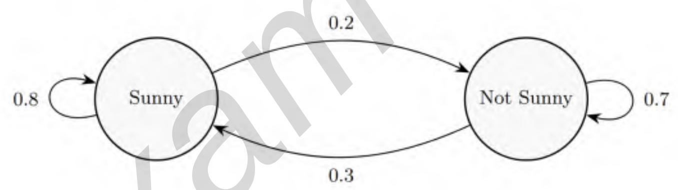
An eigenvector is $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (or any scalar multiple).

3. Diagonalization The matrix T can be diagonalized as $T = PDP^{-1}$. - (i) The matrix P is formed by the eigenvectors: $\boxed{P = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}}$ - (ii) The matrix D is the diagonal matrix of eigenvalues: $\boxed{D = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}}$

4. Long-term percentage of sunny days The long-term behavior is determined by the **steady-state vector** q, which satisfies $Tq = q$. This corresponds to the eigenvector associated with $\lambda = 1$. From part (b), the eigenvector for $\lambda = 1$ is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. To find the probability distribution, we normalize this vector so the sum of components is 1:

$$q = \frac{1}{3+2}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

The first component represents the long-term probability of a sunny day. $\boxed{60\%}$



MATHAI_21_HL_Summer_2021_Q6

Solution

The motion of the ice-skater is described by the **position vector** $\mathbf{r}(t)$:

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} ae^{bt}\cos t \\ ae^{bt}\sin t \end{pmatrix}$$

1. Velocity vector at time t The **velocity vector** $\mathbf{v}(t)$ is the derivative of the position vector with respect to time t. Applying the **product rule** to each component:

- For $x(t) = ae^{bt}\cos t$:

$$\frac{dx}{dt} = a\left(be^{bt}\cos t - e^{bt}\sin t\right) = ae^{bt}(b\cos t - \sin t)$$

- For $y(t) = ae^{bt}\sin t$:

$$\frac{dy}{dt} = a\left(be^{bt}\sin t + e^{bt}\cos t\right) = ae^{bt}(b\sin t + \cos t)$$

Thus, the velocity vector is:

$$\mathbf{v}(t) = \begin{pmatrix} ae^{bt}(b\cos t - \sin t) \\ ae^{bt}(b\sin t + \cos t) \end{pmatrix}$$

2. Magnitude of the velocity The **magnitude** of the velocity (speed) is given by $\mid \mathbf{v}(t) \mid = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$:

$$\begin{aligned} \mid \mathbf{v}(t) \mid^2 &= \left[ae^{bt}(b\cos t - \sin t)\right]^2 + \left[ae^{bt}(b\sin t + \cos t)\right]^2 \\ &= a^2e^{2bt}\left[(b^2\cos^2 t - 2b\sin t\cos t + \sin^2 t) + (b^2\sin^2 t + 2b\sin t\cos t + \cos^2 t)\right] \\ &= a^2e^{2bt}\left[b^2(\cos^2 t + \sin^2 t) + (\sin^2 t + \cos^2 t)\right] \end{aligned}$$

Using the **Pythagorean identity** $\sin^2 t + \cos^2 t = 1$:

$$\mid \mathbf{v}(t) \mid^2 = a^2e^{2bt}(b^2 + 1)$$
$$\mid \mathbf{v}(t) \mid = \sqrt{a^2e^{2bt}(1 + b^2)} = ae^{bt}\sqrt{1 + b^2}$$

3. Determination of constants a and b Given initial conditions at $t = 0$:

- Displacement $\mathbf{r}(0) = \binom{5}{0}$:

$$\begin{pmatrix} ae^0\cos 0 \\ ae^0\sin 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \implies a = 5$$

- Velocity $\mathbf{v}(0) = \binom{-3.5}{5}$:

$$\begin{pmatrix} 5e^0(b\cos 0 - \sin 0) \\ 5e^0(b\sin 0 + \cos 0) \end{pmatrix} = \begin{pmatrix} 5b \\ 5 \end{pmatrix} = \begin{pmatrix} -3.5 \\ 5 \end{pmatrix}$$

Solving for b: $5b = -3.5 \implies b = -0.7$.

Thus, $a = 5$ and $b = -0.7$.

4. Magnitude of velocity at $t = 2$ Substitute $a = 5, b = -0.7, t = 2$ into the speed formula:

$$\begin{aligned} |\,\mathbf{v}(2)\,| &= 5e^{-0.7(2)}\sqrt{1 + (-0.7)^2} \\ &= 5e^{-1.4}\sqrt{1 + 0.49} \\ &= 5e^{-1.4}\sqrt{1.49} \\ &\approx 5(0.246597)(1.220656) \approx 1.50505 \end{aligned}$$

Rounding to three significant figures, $|\,\mathbf{v}(2)\,| \approx 1.51\,\mathrm{m \cdot s^{-1}}$.

5. Distance OP when skating parallel to the y-axis The skater moves parallel to the y-axis when the x-component of the velocity is zero ($v_x = 0$):

$$\begin{aligned} ae^{bt}(b\cos t - \sin t) = 0 \\ b\cos t - \sin t = 0 \Longrightarrow \tan t = b \end{aligned}$$

Given $b = -0.7$, we find the first time $t > 0$:

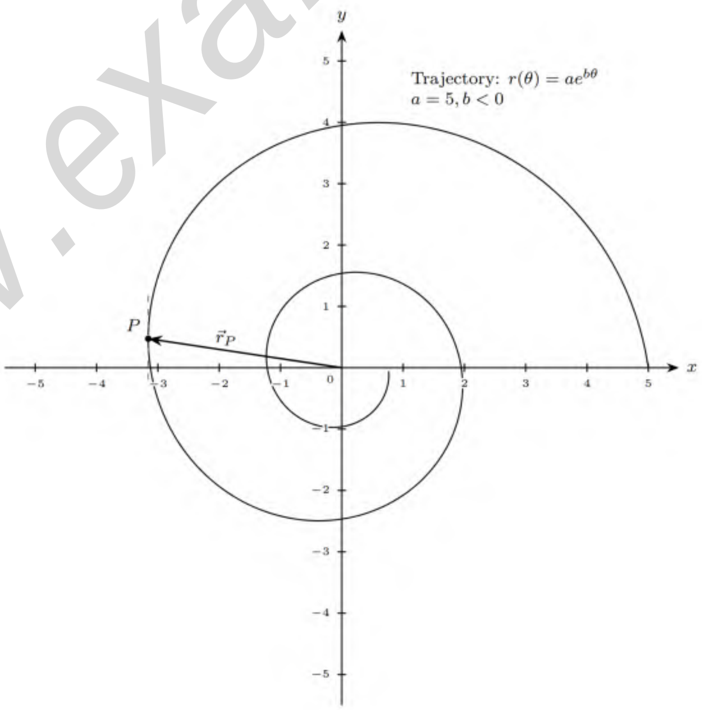
$$t = \arctan(-0.7) + \pi \approx -0.6107 + 3.1416 \approx 2.5309\,\mathrm{s}$$

The distance OP is the magnitude of the position vector $\mathbf{r}(t)$:

$$\begin{aligned} OP = |\,\mathbf{r}(t)\,| &= \sqrt{(ae^{bt}\cos t)^2 + (ae^{bt}\sin t)^2} \\ &= \sqrt{a^2e^{2bt}(\cos^2 t + \sin^2 t)} = ae^{bt} \end{aligned}$$

Substitute $a = 5, b = -0.7$, and $t \approx 2.5309$:

$$\begin{aligned} OP &= 5e^{-0.7(2.5309)} \\ &= 5e^{-1.7716} \approx 5(0.17006) \approx 0.8503 \end{aligned}$$



(a) $\mathbf{v}(t) = \begin{pmatrix} ae^{bt}(b\cos t - \sin t) \\ ae^{bt}(b\sin t + \cos t) \end{pmatrix}$
(b) $|\,\mathbf{v}(t)\,| = ae^{bt}\sqrt{1 + b^2}$

(c) $a = 5, b = -0.7$

(d) $1.51\,\mathrm{m} \cdot \mathrm{s}^{-1}$

(e) $0.850\,\mathrm{m}$

MATHAI_21_HL_Summer_2021_Q7

Solution

1. Initial Rabbit Population Growth

The initial growth of the rabbit population x is modeled by the **first-order linear differential equation**:

$$\frac{dx}{dt} = 2x$$

Given the initial condition $x(0) = 100$, we solve this by separation of variables:

$$\int \frac{1}{x}\,dx = \int 2\,dt$$
$$\ln|\ x\ | = 2t + C$$
$$x(t) = x(0)e^{2t} = 100e^{2t}$$

At $t = 1$ year:

$$\begin{aligned} x(1) &= 100e^{2(1)} \\ &= 100e^{2} \\ &\approx 738.9056 \end{aligned}$$

Rounding to the nearest whole number, the population is $\boxed{739}$ rabbits.

2. Predator-Prey Dynamics via Euler's Method

When foxes (y) are introduced, the system follows the **Lotka-Volterra equations**:

$$\frac{dx}{dt} = f(x, y) = x(2 - 0.01y)$$
$$\frac{dy}{dt} = g(x, y) = y(0.0002x - 0.8)$$

Initial conditions at $t = 0$ (introduction of foxes): $x_0 = 1000$, $y_0 = 100$. We use **Euler's method** with step size $h = 0.25$ to find populations at $t = 1$.

- **Step 1 ($t = 0$ to $t = 0.25$):**

$$x_1 = x_0 + h \cdot f(x_0, y_0) = 1000 + 0.25[1000(2 - 0.01(100))] = 1000 + 250 = 1250$$
$$y_1 = y_0 + h \cdot g(x_0, y_0) = 100 + 0.25[100(0.0002(1000) - 0.8)] = 100 - 15 = 85$$

- **Step 2 ($t = 0.25$ to $t = 0.50$):**

$$x_2 = 1250 + 0.25[1250(2 - 0.01(85))] = 1250 + 359.375 = 1609.375$$
$$y_2 = 85 + 0.25[85(0.0002(1250) - 0.8)] = 85 - 11.6875 = 73.3125$$

- **Step 3 ($t = 0.50$ to $t = 0.75$):**

$$x_3 = 1609.375 + 0.25[1609.375(2 - 0.01(73.3125))] \approx 2119.332$$
$$y_3 = 73.3125 + 0.25[73.3125(0.0002(1609.375) - 0.8)] \approx 64.549$$

- **Step 4 ($t = 0.75$ to $t = 1.00$):**

$$x_4 = 2119.332 + 0.25[2119.332(2 - 0.01(64.549))] \approx 2837.105$$
$$y_4 = 64.549 + 0.25[64.549(0.0002(2119.332) - 0.8)] \approx 58.480$$

(i) Population of rabbits: $\boxed{2837}$

(ii) Population of foxes: $\boxed{58}$

3. Phase Portrait Analysis

[Visualization]

- **(i) Point A:** At this point, the trajectory has a vertical tangent, meaning $dx/dt = 0$. The rabbit population has reached its maximum and is about to decrease, while the fox population is increasing.
- **(ii) Point B:** At this point, the trajectory has a horizontal tangent, meaning $dy/dt = 0$. The fox population has reached its maximum and is about to decrease, while the rabbit population is decreasing.

4. Non-zero Equilibrium Point

The **equilibrium point** occurs where the rates of change for both populations are zero:

$$\{x(2 - 0.01y) = 0$$
$$y(0.0002x - 0.8) = 0$$

For a non-zero solution ($x \neq 0, y \neq 0$):

$$2 - 0.01y = 0 \Longrightarrow y = \frac{2}{0.01} = 200$$
$$0.0002x - 0.8 = 0 \Longrightarrow x = \frac{0.8}{0.0002} = 4000$$

The non-zero equilibrium point is $\boxed{(4000, 200)}$.